

GENERALIZED KILLING EQUATIONS FOR SPINNING SPACES AND THE ROLE OF KILLING-YANO TENSORS

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Abstract

The generalized Killing equations for the configuration space of spinning particles (spinning space) are analysed. Solutions of these equations are expressed in terms of Killing-Yano tensors. In general the constants of motion can be seen as extensions of those from the scalar case or new ones depending on the Grassmann-valued spin variables.

Spinning particles, such as Dirac fermions, can be described by pseudo-classical mechanics models involving anticommuting c-numbers for the spin-degrees of freedom. The configuration space of spinning particles (spinning space) is an extension of an ordinary Riemannian manifold, parametrized by local coordinates $\{x^\mu\}$, to a graded manifold parametrized by local coordinates $\{x^\mu, \psi^\mu\}$, with the first set of variables being Grassmann-even (commuting) and the second set Grassmann-odd (anticommuting) [1-3].

The equation of motion of a spinning particle on a geodesic is derived from the action:

$$S = \int d\tau \left(\frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x) \psi^\mu \frac{D\psi^\nu}{D\tau} \right). \quad (1)$$

The corresponding world-line hamiltonian is given by:

$$H = \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu \quad (2)$$

where $\Pi_\mu = g_{\mu\nu} \dot{x}^\nu$ is the covariant momentum.

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For any constant of motion $\mathcal{J}(x, \Pi, \psi)$, the bracket with H vanishes

$$\{H, \mathcal{J}\} = 0. \quad (3)$$

If we expand $\mathcal{J}(x, \Pi, \psi)$ in a power series in the canonical momentum

$$\mathcal{J} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{J}^{(n)\mu_1 \dots \mu_n}(x, \psi) \Pi_{\mu_1} \dots \Pi_{\mu_n} \quad (4)$$

then the bracket $\{H, \mathcal{J}\}$ vanishes for arbitrary Π_{μ} if and only if the components of \mathcal{J} satisfy the generalized Killing equations [1] :

$$\mathcal{J}_{(\mu_1 \dots \mu_n; \mu_{n+1})}^{(n)} + \frac{\partial \mathcal{J}_{(\mu_1 \dots \mu_n}^{(n)}}{\partial \psi^{\sigma}} \Gamma_{\mu_{n+1})\lambda}^{\sigma} \psi^{\lambda} = \frac{i}{2} \psi^{\rho} \psi^{\sigma} R_{\rho\sigma\nu(\mu_{n+1}} \mathcal{J}^{(n+1)\nu}_{\mu_1 \dots \mu_n)} \quad (5)$$

where the parentheses denote full symmetrization over the indices enclosed.

In general the symmetries of a spinning-particle model can be divided into two classes. First, there are four independent *generic* symmetries which exist in any theory [1-3] :

1. Proper-time translations generated by the hamiltonian H (2)
2. Supersymmetry generated by the supercharge

$$Q = \Pi_{\mu} \psi^{\mu} \quad (6)$$

3. Chiral symmetry generated by the chiral charge

$$\Gamma_{*} = \frac{i^{[\frac{d}{2}]}}{d!} \sqrt{g} \epsilon_{\mu_1 \dots \mu_d} \psi^{\mu_1} \dots \psi^{\mu_d} \quad (7)$$

4. Dual supersymmetry, generated by the dual supercharge

$$Q^{*} = i\{\Gamma_{*}, Q_0\} = \frac{i^{[\frac{d}{2}]}}{(d-1)!} \sqrt{g} \epsilon_{\mu_1 \dots \mu_d} \Pi^{\mu_1} \psi^{\mu_2} \dots \psi^{\mu_d} \quad (8)$$

where d is the dimension of space-time.

The second kind of conserved quantities, called *non-generic*, depend on the explicit form of the metric $g_{\mu\nu}(x)$. In the recent literature there are exhibited the constants of motion in the Schwarzschild [4], Taub-NUT [5-7], Kerr-Newman [3] spinning spaces.

In what follows we shall deal with the *non-generic* constants of motion in connection with the Killing equations (5) looking for the general features of the solutions. In general the constants of motion can be seen as extensions of the constants from the scalar case or new ones depending on the Grassmann-valued spin variables $\{\psi^{\mu}\}$.

Let us assume that the number of terms in the series (4) is finite. That means that, for a given n , $\mathcal{J}_{(\mu_1 \dots \mu_{n+1})}^{(n+1)}$ vanishes and the last non-trivial equation from the system of Killing equations (5) becomes homogeneous :

$$\mathcal{J}_{(\mu_1 \dots \mu_n; \mu_{n+1})}^{(n)} + \frac{\partial \mathcal{J}_{(\mu_1 \dots \mu_n)}^{(n)}}{\partial \psi^\sigma} \Gamma_{\mu_{n+1} \lambda}^\sigma \psi^\lambda = 0. \quad (9)$$

In order to solve the system of coupled differential equations (5) one starts with a $\mathcal{J}_{\mu_1 \dots \mu_n}^{(n)}$ solution of the homogeneous equation (9). This solution is introduced in the right-hand side (RHS) of the generalized Killing equation (5) for $\mathcal{J}_{\mu_1 \dots \mu_{n-1}}^{(n-1)}$ and the iteration is carried on to $n = 0$.

In fact, for the bosonic sector, neglecting the Grassmann variables $\{\psi^\mu\}$, all the generalized Killing equations (5) are homogeneous and decoupled. The first equation shows that $\mathcal{J}^{(0)}$ is a trivial constant, the next one is the equation for the Killing vectors and so on. In general, the homogeneous equation (9) for a given n in which all spin degrees of freedom are neglected, defines a Killing tensor of valence n

$$\mathcal{J}_{\mu_1 \dots \mu_n; \mu_{n+1}}^{(n)} = 0 \quad (10)$$

and

$$\mathcal{J} = \mathcal{J}_{\mu_1 \dots \mu_n}^{(n)} \Pi^{\mu_1} \dots \Pi^{\mu_n} \quad (11)$$

is a first integral of the geodesic equation [8].

For the spinning particles, even if one starts with a Killing tensor of valence n , solution of eq.(10) in which all spin degrees of freedom are neglected, the components $\mathcal{J}_{\mu_1 \dots \mu_m}^{(m)}$ ($m < n$) will receive a nontrivial spin contribution.

Therefore the quantity (11) is no more conserved and the actual constant of motion is

$$\mathcal{J} = \sum_{m=0}^n \frac{1}{m!} \mathcal{J}^{(m)}_{\mu_1 \dots \mu_m} \Pi_{\mu_1} \dots \Pi_{\mu_m} \quad (12)$$

in which $\mathcal{J}^{(m)}_{\mu_1 \dots \mu_m}$ with $m < n$ has a nontrivial spin-dependent expression.

We shall illustrate the above construction with a few examples. Since for $n = 0$ eq. (10) is trivial, we shall consider the next case, namely $n = 1$. In this case eq. (10) is satisfied by a Killing vector R_μ

$$R_{(\mu; \nu)} = 0. \quad (13)$$

Introducing this Killing vector in the RHS of the generalized Killing equation (5) for $n = 0$ one obtains for the Killing scalar [7]

$$\mathcal{J}^{(0)} = \frac{i}{2} R_{[\mu; \nu]} \psi^\mu \psi^\nu \quad (14)$$

where the square bracket denotes antisymmetrization with norm one.

A more involved example is given by a Killing tensor $K_{\mu\nu}$ satisfying equation (10) for $n = 2$:

$$K_{(\mu\nu;\lambda)} = 0. \quad (15)$$

Unfortunately it is not possible to find a closed, analytic expression for the spin corrections to the quantity (11) in terms of the components of the Killing tensor $K_{\mu\nu}$ and its derivatives. But assuming that the Killing tensor $K_{\mu\nu}$ can be written as a symmetrized product of two Killing-Yano tensors, the construction of the conserved quantity (12) is feasible. We remind that a tensor $f_{\mu_1 \dots \mu_r}$ is called Killing-Yano of valence r if it is totally antisymmetric and satisfies the equation [9]

$$f_{\mu_1 \dots \mu_{r-1}(\mu_r;\lambda)} = 0. \quad (16)$$

For the generality, let us assume that the Killing tensor can be written as a symmetrized product of two different Killing-Yano tensors

$$K_{ij}^{\mu\nu} = \frac{1}{2}(f_i^\mu f_j^{\nu\lambda} + f_i^\nu f_j^{\mu\lambda}) \quad (17)$$

where $f_i^{\mu\nu}$ is a Killing Yano tensor of type i and the Killing tensor has two additional indices i, j to record the fact that it is formed from two different Killing-Yano tensors ($i \neq j$).

Introducing the Killing tensor (17) in the RHS of eq.(5) for $n = 1$ we get a spin contribution to the Killing vector [3]:

$$I_{ij}^{(1)\mu} = \frac{i}{2}\Psi^\lambda\Psi^\sigma(f_i^\nu{}_\sigma D_\nu f_j^\mu{}_\lambda + f_j^\nu{}_\sigma D_\nu f_i^\mu{}_\lambda + \frac{1}{2}f_i^{\mu\rho}c_{j\lambda\sigma\rho} + f_j^{\mu\rho}c_{i\lambda\sigma\rho}) \quad (18)$$

and using this quantity in the RHS of eq.(5) for $n = 0$ we get for the Killing scalar

$$J_{ij}^{(0)} = -\frac{1}{4}\Psi^\lambda\Psi^\sigma\Psi^\rho\Psi^\tau(R_{\mu\nu\lambda\sigma}f_i^\mu{}_\rho f_j^\nu{}_\tau + \frac{1}{2}c_{i\lambda\sigma}{}^\pi c_{j\rho\tau\pi}) \quad (19)$$

where the tensor $c_{i\mu\nu\lambda}$ is [3] :

$$c_{i\mu\nu\lambda} = -2f_{i[\nu\lambda;\mu]}. \quad (20)$$

Higher orders of the generalized equations (5) can be treated similarly, but the corresponding expressions are quite involved.

In what follows we shall analyze the homogeneous equation (9) looking for solutions depending on the Grassmann variables $\{\psi^\mu\}$. Even the lowest order equation with $n = 0$ has a nontrivial solution [7,10]

$$\mathcal{J}^{(0)} = \frac{i}{4}f_{\mu\nu}\psi^\mu\psi^\nu \quad (21)$$

where $f_{\mu\nu}$ is a Killing-Yano tensor covariantly constant. Moreover $\mathcal{J}^{(0)}$ is a separately conserved quantity.

Going to the next equation (9) with $n = 1$, a natural solution is:

$$\mathcal{J}_\mu^{(1)} = R_\mu f_{\lambda\sigma} \psi^\lambda \psi^\sigma \quad (22)$$

where R_μ is a Killing vector and again $f_{\lambda\sigma}$ is a Killing-Yano tensor covariantly constant. Introducing this solution in the RHS of the eq. (5) with $n = 0$, after some calculations, we get for $\mathcal{J}^{(0)}$ [10]:

$$\mathcal{J}^{(0)} = \frac{i}{2} R_{[\mu;\nu]} f_{\lambda\sigma} \psi^\mu \psi^\nu \psi^\lambda \psi^\sigma \quad (23)$$

Combining eq.(22) and (23) with the aid of eqs.(12) we get a constant of motion which is peculiar to the spinning case:

$$\mathcal{J} = f_{\mu\nu} \psi^\mu \psi^\nu \left(R_\lambda \Pi^\lambda + \frac{i}{2} R_{[\lambda;\sigma]} \psi^\lambda \psi^\sigma \right). \quad (24)$$

In fact this constant of motion is not completely new and it can be expressed in terms of the quantities (14) and (21).

Another ψ -dependent solution of eq.(9) for $n = 1$ can be generated from a Killing-Yano tensor of valence r :

$$\mathcal{J}_{\mu_1}^{(1)} = f_{\mu_1 \mu_2 \dots \mu_r} \psi^{\mu_2} \dots \psi^{\mu_r}. \quad (25)$$

Again introducing this quantity in the RHS of eq.(5) for $n = 0$ we get for $\mathcal{J}^{(0)}$:

$$\mathcal{J}^{(0)} = \frac{i}{r+1} (-1)^{r+1} f_{[\mu_1 \dots \mu_r; \mu_{r+1}]} \psi^{\mu_1} \dots \psi^{\mu_{r+1}} \quad (26)$$

and the constant of motion corresponding to these solutions of the Killing equations is [10]:

$$Q_f = f_{\mu_1 \dots \mu_r} \Pi^{\mu_1} \psi^{\mu_2} \dots \psi^{\mu_r} + \frac{i}{r+1} (-1)^{r+1} f_{[\mu_1 \dots \mu_r; \mu_{r+1}]} \psi^{\mu_1} \dots \psi^{\mu_{r+1}}. \quad (27)$$

Therefore the existence of a Killing-Yano tensor of valence r is equivalent to the existence of a supersymmetry for the spinning space with supercharge Q_f which anticommutes with Q_0 . A similar result was obtained in ref.[11] in which it is discussed the role of the generalized Killing-Yano tensors, with the framework extended to include electromagnetic interactions.

The aim of this paper was to point out the important role of the Killing-Yano tensors to generate solutions of the generalized Killing equations. This aspect is closely connected with the fact that the Killing-Yano tensors can be understood as objects generating *non-generic* supersymmetries [3]. For the first orders of eqs.(5) and (9) ($n \leq 2$), which are usually encountered in theories of interest, we presented the complete form of the solutions. With some ability, it is possible to investigate the higher orders of the generalized Killing equations, but it seems that one cannot go much far with simple, transparent expressions. The extension of these results for the motion of spinning particles in spaces with torsion and/or in the presence of an electromagnetic field will be discussed elsewhere.

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References

- [1] See e.g. R.H.Rietdijk, "*Applications of supersymmetric quantum mechanics*", Ph.D.Thesis, Univ. Amsterdam (1992), and references therein.
- [2] R.H.Rietdijk and J.W.van Holten, *Class.Quant.Grav.* **7** (1990) 247.
- [3] G.W.Gibbons, R.H.Rietdijk and J.W.van Holten, *Nucl.Phys.* **B404** (1993) 42.
- [4] R.H.Rietdijk and J.W.van Holten, *Clas.Quant.Grav.* **10** (1993) 375.
- [5] M.Visinescu, *Class.Quant.Grav.* **11** (1994) 1867.
- [6] M.Visinescu, *Phys.Lett.* **B339** (1994) 28.
- [7] J.W.van Holten, *Phys.Lett.* **B342** (1995) 47.
- [8] W.Dietz and R.Rüdinger, *Proc.R.Soc.London* **A375** (1981) 361.
- [9] K.Yano, *Ann.Math* **55** (1952) 328.
- [10] D.Vaman and M.Visinescu, *Phys.Rev.* **D54** (1996) 1398.
- [11] M.Tanimoto, *Nucl.Phys.* **B442** (1995) 549.